Mathematical Statistics Qualifier Examination
Part I of the STAT AREA EXAM
May 25, 2022; 9:00 AM - 11:00 AM

There are 4 problems. You are required to solve them all. Show detailed work for full credit.

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

NAME: ______________________________ **ID:** ______________________________

Signature: ______________________________

1. Let X_i, $i = 1, \ldots, n$ be independently distributed as $N(\alpha + \beta t_i, \sigma^2)$, where α, β, and σ^2 are unknown, and the t's are known constants that are not all equal.

   (a) Find the complete sufficient statistics for $\left(\alpha, \beta, \sigma^2\right)$.

   (b) Find the maximum likelihood estimators of α and β.

   (c) Show that the MLEs obtained in part (b) are UMVUEs of α and β.

2. Suppose that $X_1, \ldots, X_n$ is a random sample from the exponential distribution with pdf

$$f(x|\theta) = \theta e^{-\theta x}, \quad x > 0, \ \theta > 0.$$

Find the UMVUE of $e^{-k\theta}$ for fixed $k > 0$.
Hint: Define

$$U = \begin{cases} 1, & X_1 > k \\ 0, & X_1 \leq k. \end{cases}$$

3. Let $X_1, \cdots, X_5$ be a random sample from a distribution with pdf $f(x;\theta) = \theta^x(1-\theta)^{1-x},\ x = 0, 1, \cdots$. Consider testing H_0: $\theta = 0.5$ versus H_1: $\theta < 0.5$.

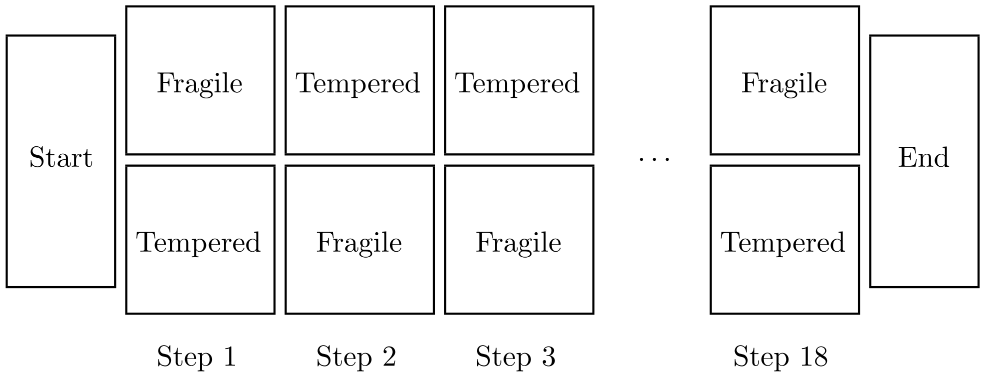
   (a) Find the UMP test.

   (b) Find the significance level when we reject H_0 if $\sum_{i=1}^{5} X_i < 1$.

   (c) Find the significance level when we reject H_0 if $\sum_{i=1}^{5} X_i < 0$.

   (d) By using the randomized test, modify the tests given in parts (b) and (c) to find a test with significance level $\alpha = 1/8$.

4. Squid Game is a television series consisting of several survival games released in 2021 by Netflix. The fifth game in this TV series is Glass Stepping Stones. Players have to jump their way through a bridge consisting of 18 pairs of glass squares. For each pair of the glass squares laid side by side, one glass square is tempered and will hold up to two players at a time, and the square beside it is fragile and whoever jumps on it falls to their death. The tempered glass and fragile glasses are assigned randomly in each pair. The following figure shows a possible arrangement.



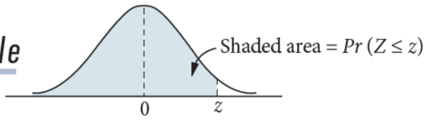
Sixteen players numbered 1 through 16 are participating in this game. Player 1 starts crossing the bridge, and Player i follows Player $i-1$, $i = 2, \cdots, 16$ by advancing to the pair of stepping glasses right behind Player i. A player has to make a blind guess between the two glasses when this person is the first one to advance to the next pair of stepping glasses. With this game, each of the 16 players does not have equal advantage. The first player has no prior knowledge of which side to take, whereas the last player has seen 15 players make the right or wrong decision, making their challenge simply to get over the bridge in time. Three players survived at the end of the game. Find the probability that at least 3 players survive after this game. A standard normal distribution table is given on the last page, and the values of square roots are given below. For example, $\sqrt{8.7} = 2.95$ according to this table. (Hint: You can use the Central limit Theorem. Continuity correction is required.)

Table: $\sqrt{a}$

a	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.00	0.32	0.45	0.55	0.63	0.71	0.77	0.84	0.89	0.95
1	1.00	1.05	1.10	1.14	1.18	1.22	1.26	1.30	1.34	1.38
2	1.41	1.45	1.48	1.52	1.55	1.58	1.61	1.64	1.67	1.70
3	1.73	1.76	1.79	1.82	1.84	1.87	1.90	1.92	1.95	1.97
4	2.00	2.02	2.05	2.07	2.10	2.12	2.14	2.17	2.19	2.21
5	2.24	2.26	2.28	2.30	2.32	2.35	2.37	2.39	2.41	2.43
6	2.45	2.47	2.49	2.51	2.53	2.55	2.57	2.59	2.61	2.63
7	2.65	2.66	2.68	2.70	2.72	2.74	2.76	2.77	2.79	2.81
8	2.83	2.85	2.86	2.88	2.90	2.92	2.93	2.95	2.97	2.98
9	3.00	3.02	3.03	3.05	3.66	3.08	3.10	3.11	3.13	3.15

Answer to Question 4:

Table A.1: Standard Normal Distribution Table



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

z	Area
3.50	0.99976737
4.00	0.99996833
4.50	0.99999660
5.00	0.99999971